

Spectral Representation and the Averaging Problem in Cosmology ¹

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Abstract

We investigate the averaging problem in cosmology as the problem of introducing a distance between spaces.

We first introduce the spectral distance, which is a measure of closeness between spaces defined in terms of the spectra of the Laplacian. Then we define \mathcal{S}_N , a space of all spaces equipped with the spectral distance. We argue that \mathcal{S}_N can be regarded as a metric space and that it also possess other desirable properties. These facts make \mathcal{S}_N a suitable arena for spacetime physics.

We apply the spectral framework to the averaging problem: We sketch the model-fitting procedure in terms of the spectral representation, and also discuss briefly how to analyze the dynamical aspects of the averaging procedure with this scheme. In these analyses, we are naturally led to the concept of the apparatus- and the scale-dependent effective evolution of the universe. These observations suggest that the spectral scheme seems to be suitable for the quantitative analysis of the averaging problem in cosmology.

1 Introduction

The *averaging problem* is one of the fundamental problems in cosmology that we have not yet understood sufficiently so far[1, 2].

It can be summarized as follows: In cosmology we want to understand the whole picture of our universe. However since the structure of the universe is so complicated that we can understand it only with the help of some format of recognition, viz. cosmological models. Thus, cosmology in principle requires a mapping procedure from reality to a model. Alternatively, one may regard this procedure as the procedure of averaging the real, complicated geometry in some

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manner in order to assign a simplified model-geometry to it. Now the problem is that Einstein equation is highly nonlinear, so that the effective dynamics of the averaged spatial geometry is expected to be highly complicated. Moreover, the effective dynamics of the averaged spatial geometry would not in general match the dynamics of the assigned model. Hence we should first analyze and understand the averaging procedure *itself*, otherwise we would make serious mistakes in finding out the evolution of the universe and/or matter content of the universe. If we state symbolically, the averaging procedure and the Einstein equation do not commute with each other.

Once we start investigating the averaging procedure itself, we immediately face with the trouble that we do not have a suitable ‘language’ for describing it. In order to formulate the approximation of the real geometry by a certain model, the concepts like ‘closeness’ or ‘distance’ between spaces are indispensable. However there has been no established mathematical theory so far which deals with these concepts and which can be applied to spacetime physics³. Here we will study the averaging problem in cosmology as *the problem of defining a distance between spaces*.

For this purpose we would like to focus on the *spectral representation* of spatial geometry [5] as a promising attempt in this direction. The basic idea of the spectral representation is simple: We utilize the ‘sound’ of a space to characterize the geometrical structures of the space. This idea immediately reminds us of a famous problem in mathematics, ‘Can one hear the shape of a drum?’ [6] In imitation of this phrase, we can state that ‘Let us hear the shape of the universe!’ [5]

2 The spectral distance

We should materialize the idea of ‘hearing the shape of the universe’ in a definite form. For definiteness we confine ourselves to $(D - 1)$ -dimensional Riemannian manifolds that are spatial (metric signature $(+, \dots, +)$) and compact without boundaries. Let *Riem* denote this class of Riemannian manifolds. Now we set up the eigenvalue problem of the Laplace-Beltrami operator, $\Delta f = -\lambda f$. Then we obtain the *spectra*, viz. the set of eigenvalues (numbered in increasing order), $\{\lambda_n\}_{n=0}^{\infty}$. We note that, on dimensional grounds, the lower (higher) spectrum corresponds to the larger (smaller) scale behavior of geometry.

Suppose we want to compare two geometries \mathcal{G} and \mathcal{G}' . Our strategy is hence to compare the spectra $\{\lambda_n\}_{n=1}^N$ for \mathcal{G} with the spectra $\{\lambda'_n\}_{n=1}^N$ for \mathcal{G}' . However, taking a difference $\lambda'_n - \lambda_n$ simply is not appropriate for our purpose: The simple

³ As a concept which one should recall in this context, there is the Gromov-Hausdorff distance $d_{GH}(X, Y)$ between two compact metric spaces [3, 4]. Though it plays a central role in the convergence theory of Riemannian geometry, its abstract nature may be a big obstacle for its effective application to spacetime physics.

difference $\lambda'_n - \lambda_n$ would in general count the difference in the higher spectra (corresponding to the smaller scale behavior of geometry) with more weight. In spacetime physics, however, the difference in the larger scale behavior of geometry is of more importance than the one in the smaller scale behavior of geometry. (This is the precise description of the ‘spacetime foam picture’ and the scale-dependent topology [7, 5].) In addition, the difference $\lambda'_n - \lambda_n$ has a physical dimension $[\text{Length}^{-2}]$, which is not very comfortable either. Hence, we should rather take the ratio $\frac{\lambda'_n}{\lambda_n}$; Then the difference $\delta\lambda_n := \lambda'_n - \lambda_n$ in the lower spectrum is counted with more weight as $\frac{\lambda'_n}{\lambda_n} = 1 + \frac{\delta\lambda_n}{\lambda_n}$.

Now a measure of closeness $d_N(\mathcal{G}, \mathcal{G}')$ between two geometries \mathcal{G} and \mathcal{G}' can be introduced by comparing the spectra $\{\lambda_n\}_{n=1}^N$ (for \mathcal{G}) with $\{\lambda'_n\}_{n=1}^N$ (for \mathcal{G}') as

$$d_N(\mathcal{G}, \mathcal{G}') = \sum_{n=1}^N \mathcal{F}(\lambda'_n/\lambda_n) \quad . \quad (1)$$

Here the zero mode $\lambda_0 = \lambda'_0 = 0$ is not included in the summation, and N is the cut-off number which can be treated as a running parameter. The function $\mathcal{F}(x)$ ($x > 0$) is a suitably chosen function which satisfies $\mathcal{F} \geq 0$, $\mathcal{F}(1) = 0$, $\mathcal{F}(1/x) = \mathcal{F}(x)$, and $\mathcal{F}(y) > \mathcal{F}(x)$ if $y > x \geq 1$. We also note that the cut-off number N characterizes up to which scale two geometries \mathcal{G} and \mathcal{G}' are compared. In this way, $d_N(\mathcal{G}, \mathcal{G}')$ is suitable for the scale-dependent description of the geometry.

At this stage, some comments may be appropriate on the spectral representation in general. It is true that the spectra can be explicitly calculated only for restricted cases. However, still there are several advantages for the spectral representation. First, the concept of the spectra itself is very clear. This is important for practical applications in physics. Second, even when the exact spectra themselves are not known explicitly, the perturbation analysis gives us important information on the spectra. For instance, one can investigate the perturbed spectra around some well-understood spectra, just like one investigates the perturbed metric around the Minkowski metric. Third, in spacetime physics, the lower spectra are more important than the higher spectra, since the former spectra reflect the large scale structure of the universe. Thus, even a few lower-lying spectra, which are in general easier to compute than the higher spectra, carry important information. (For more details, see Ref. [5].)

We note that the property $\mathcal{F}(\lambda'_n/\lambda_n) \rightarrow 0$ as $n \rightarrow \infty$ is required for the convergence of d_N as $N \rightarrow \infty$. Thus, it follows that $\lambda'_n/\lambda_n \rightarrow 1$ as $n \rightarrow \infty$ should hold for convergence. Combined with the Weyl’s asymptotic formula [6, 8], it means that, in the $N \rightarrow \infty$ limit, the dimension and the volume of \mathcal{G} and \mathcal{G}' should be same in order to give a finite $d_N(\mathcal{G}, \mathcal{G}')$ as $N \rightarrow \infty$ [5]. When N is kept finite as most of the cases we consider, these conditions need not necessarily to be satisfied for a finite d_N .

In order to utilize the measure $d_N(\mathcal{G}, \mathcal{G}')$ efficiently, it is desirable that $d_N(\mathcal{G}, \mathcal{G}')$ satisfies the axioms of distance, or at least some modified version of them:

- (I) *Positivity*: $d_N(\mathcal{G}, \mathcal{G}') \geq 0$, and $d_N(\mathcal{G}, \mathcal{G}') = 0 \Leftrightarrow \mathcal{G} \sim \mathcal{G}'$, where \sim means equivalent in the sense of isospectral manifolds [6, 8],
- (II) *Symmetry*: $d_N(\mathcal{G}, \mathcal{G}') = d_N(\mathcal{G}', \mathcal{G})$,
- (III) *Triangle Inequality*: $d_N(\mathcal{G}, \mathcal{G}') + d_N(\mathcal{G}', \mathcal{G}'') \geq d_N(\mathcal{G}, \mathcal{G}'')$.

Among several possibilities for the choice of $\mathcal{F}(x)$, there is one very important choice:

- (a) $\mathcal{F}_a(x) = \frac{1}{2} \ln \frac{1}{2}(\sqrt{x} + 1/\sqrt{x})$.
Then Eq.(1) becomes [5]

$$d_N(\mathcal{G}, \mathcal{G}') = \frac{1}{2} \sum_{n=1}^N \ln \frac{1}{2} \left(\sqrt{\frac{\lambda'_n}{\lambda_n}} + \sqrt{\frac{\lambda_n}{\lambda'_n}} \right) . \quad (2)$$

It is notable that this form for d_N can be related to the reduced density matrix element in quantum cosmology under some circumstances [5]. Namely, a long (short) spectral distance $d_N(\mathcal{G}, \mathcal{G}')$ can be interpreted as a strong (weak) quantum decoherence between \mathcal{G} and \mathcal{G}' for some cases in quantum cosmology. This interpretation of d_N gives one motivation for the choice of $\mathcal{F}_a(x)$.

The measure of closeness d_N defined in Eq.(2) satisfies (I) and (II) of the distance axioms, but it does not satisfy the triangle inequality (III) [5]. Significantly enough, however, the failure of the triangle inequality turns out to be only a mild one since a universal constant $c(> 0)$ can be chosen such that $d'_N(\mathcal{G}, \mathcal{G}') := d_N(\mathcal{G}, \mathcal{G}') + c$ recovers the triangle inequality [9, 10]. Here c is universal in the sense that c can be chosen independent of \mathcal{G} , \mathcal{G}' and \mathcal{G}'' although it depends on N . This fact leads to a significant consequence below: d_N and its modification \bar{d}_N (see below) are closely related to each other, which helps us reveal the nice properties of d_N .

There is another important choice for $\mathcal{F}(x)$:

- (b) $\mathcal{F}_b(x) := \frac{1}{2} \ln \max(\sqrt{x}, 1/\sqrt{x})$.

This is a slight modification of \mathcal{F}_a . In this case, Eq.(1) becomes

$$\bar{d}_N(\mathcal{G}, \mathcal{G}') = \frac{1}{2} \sum_{n=1}^N \ln \max \left(\sqrt{\frac{\lambda'_n}{\lambda_n}}, \sqrt{\frac{\lambda_n}{\lambda'_n}} \right) . \quad (3)$$

Note that \bar{d}_N satisfies (I)-(III), so that it is a distance.

Now, each measure of closeness introduced above has its own advantage: d_N in (a) has an analytically neat form and it can be related to the quantum decoherence between \mathcal{G} and \mathcal{G}' in the context of quantum cosmology; However, it

does not satisfy the triangle inequality. On the other hand, the measure \bar{d}_N in (b) is a distance in a rigorous sense, although its form is not very convenient for practical applications (it contains *max*). Quite surprisingly, it turns out that d_N and \bar{d}_N are deeply related to each other. To discuss this property, we introduce an r -ball centered at \mathcal{G} defined by d_N in Eq.(2),

$$B(\mathcal{G}, r; d_N) := \{\mathcal{G}' \in \text{Riem}/\sim \mid d_N(\mathcal{G}, \mathcal{G}') < r\} \quad .$$

Here \sim indicates the identification of isospectral manifolds. In the same manner, we also introduce an r -ball centered at \mathcal{G} defined by \bar{d}_N , $B(\mathcal{G}, r; \bar{d}_N)$.

Now we can show that [10]

Theorem 1

The set of balls $\{B(\mathcal{G}, r; d_N) \mid \mathcal{G} \in \text{Riem}/\sim, r > 0\}$ and the set of balls $\{B(\mathcal{G}, r; \bar{d}_N) \mid \mathcal{G} \in \text{Riem}/\sim, r > 0\}$ generate the same topology on Riem/\sim .

For the proof of *Theorem 1*, first we should show that the set of all balls $\{B(\mathcal{G}, r; d_N) \mid \mathcal{G} \in \text{Riem}/\sim, r > 0\}$ can actually define a topology (let us call it “ d_N -topology”), viz. the set of all balls can be a basis of open sets. This property is far from trivial, because of the failure of the triangle inequality for d_N .⁴ Next, we need to show that any ball defined by d_N (resp. \bar{d}_N) is an open set in \bar{d}_N -topology (resp. d_N -topology) [10].

From *Theorem 1*, we immediately obtain

Corollary

The space $\mathcal{S}_N^o := (\text{Riem}, d_N)/\sim$ is a metrizable space. The distance function for metrization is provided by \bar{d}_N .

Hence we can extend \mathcal{S}_N^o to its completion⁵, \mathcal{S}_N . Due to *Theorem 1* and its *Corollary*, it is justified to treat d_N as a distance and to regard \mathcal{S}_N as a metric space, provided that we resort to the distance function \bar{d}_N whenever the triangle inequality is needed in the arguments.

We can also show that \mathcal{S}_N has several other desirable properties [10]:

Theorem 2

The space \mathcal{S}_N is paracompact.

Corollary

There exists partition of unity subject to any open covering of \mathcal{S}_N .

⁴ On the other hand, a similar set of balls defined by \bar{d}_N can define a topology (let us call it \bar{d}_N -topology), since \bar{d}_N is a distance.

⁵ It is desirable to investigate the structure of \mathcal{S}_N intensively as a purely mathematical object.

Theorem 3

The space \mathcal{S}_N is locally compact.

Due to this property, we can construct an integral over \mathcal{S}_N [11], which is essential to consider, e.g., probability distributions over \mathcal{S}_N .

Corollary

If a sequence of continuous functions on \mathcal{S}_N , $\{f_n\}_{n=1}^\infty$, pointwise converges to a function f_∞ , then f_∞ is continuous on a dense subset of \mathcal{S}_N .

Theorem 4

The space \mathcal{S}_N satisfies the second countability axiom.

These properties of \mathcal{S}_N suggest that the space \mathcal{S}_N can serve as a basic arena for spacetime physics. From now on we call d_N in Eq.(1) (the form of d_N in Eq.(2) in particular) a *spectral distance* for brevity.

3 Model-fitting procedure in cosmology

Now let us come back to the averaging problem in cosmology. Regarding this problem, there are several underlying issues as follows:

- (1) How to select out a time-slicing for a given spacetime ('reality'), which in general would possess no symmetry.

Furthermore,

- (2) How to incorporate the spatial diffeomorphism invariance,
- (3) How to incorporate the scale-dependent aspects of the geometrical structures, and
- (4) How to incorporate the apparatus dependence of the observed information

to the averaging procedure of geometry.

Considering its several desirable properties, the spectral representation seems to serve as a suitable 'language' for formulating and analyzing these issues. In particular the space \mathcal{S}_N introduced in the previous section provides an appropriate platform for these discussions.

As a demonstration, let us give a rough sketch of the mapping procedure from reality to a model in terms of the spectral representation.

We here consider how to assign a model spacetime to a given spacetime ('reality'). We first fix notations.

- (1°) We consider a portion of a spacetime (\mathcal{M}, g) bounded by two non-intersecting spatial sections Σ_0 and Σ_1 of (\mathcal{M}, g) . Let us denote this portion as $(\mathcal{M}, g)_{(0,1)}$.

- (2°) Let $Slice_o(\mathcal{M}, g)_{(0,1)}$ be a set of all possible time-slicings of $(\mathcal{M}, g)_{(0,1)}$.
- (3°) Hence, a slice $s \in Slice_o(\mathcal{M}, g)_{(0,1)}$ can be identified with a parameterized set of spatial geometries $\{(\Sigma, h(\beta))\}_{0 \leq \beta \leq 1}$. Let $(\mathcal{M}, g)_{(0,1),s}$ denote this set for brevity.
- (4°) Let $\{models\}$ be a set of model spacetimes, bounded by two non-intersecting spatial sections Σ'_0 and Σ'_1 with a particular time-slicing, $(\mathcal{M}', g')_{(0,1),s'}$. Here, we distinguish between the identical spacetimes (\mathcal{M}', g') with different choices of two non-intersecting spatial sections (Σ'_0 and Σ'_1) and/or different choices of a time-slicing (s').

It is notable that the metric-space structure of \mathcal{S}_N induces the same structure on $Slice_o(\mathcal{M}, g)_{(0,1)}$ also: Let $s_1 := \{(\Sigma, h_1(\beta))\}_{0 \leq \beta \leq 1}$ and $s_2 := \{(\Sigma, h_2(\beta))\}_{0 \leq \beta \leq 1}$ are any elements in $Slice_o(\mathcal{M}, g)_{(0,1)}$. Then, we can define

$$D_N(s_1, s_2) := \int_0^1 [d_N((\Sigma_1, h_1(\beta)), (\Sigma_2, h_2(\beta)))] d\mu(\beta) \quad , \quad (4)$$

where $\mu(\beta)$ is a positive-definite measure. Clearly $Slice_o(\mathcal{M}, g)_{(0,1)}$ with D_N becomes a metrizable space reflecting the same property of \mathcal{S}_N . Thus, we can consider its completion, $Slice(\mathcal{M}, g)_{(0,1)}$.

Now we describe the procedure of assigning a model spacetime to reality.

[1] *The choice of time-slicing*

Let us choose and fix one model spacetime with a particular time-slicing $(\mathcal{M}', g')_{(0,1),s'}$. We can select the most suitable time-slicing of $(\mathcal{M}, g)_{(0,1)}$ w.r.t. (with respect to) the model $(\mathcal{M}', g')_{(0,1),s'}$ as follows: For each parameter β ($0 < \beta < 1$), the closeness between the slice $(\Sigma, h(\beta))$ in $(\mathcal{M}, g)_{(0,1),s}$ and the slice $(\Sigma', h'(\beta))$ in $(\mathcal{M}', g')_{(0,1),s'}$ can be measured by the spectral distance $d_{A,\Lambda}((\Sigma, h(\beta)), (\Sigma', h'(\beta)))$. Here the suffixes A and Λ indicate, respectively, the elliptic operator (we here consider the Laplacian for simplicity) and the cut-off number (viz. N in the previous section) for defining the spectral distance. Physically, A and Λ symbolize the observational apparatus and the cut-off scale, respectively.

Now we can select out the most suitable time-slicing of $(\mathcal{M}, g)_{(0,1)}$ w.r.t. the model $(\mathcal{M}', g')_{(0,1),s'}$ as

Select $s_0 \in Slice(\mathcal{M}, g)_{(0,1)}$ **s.t.**

$$D_{A,\Lambda} \left((\mathcal{M}, g)_{(0,1),s}, (\mathcal{M}', g')_{(0,1),s'} \right) \\ := \int_0^1 [d_{A,\Lambda} ((\Sigma, h(\beta)), (\Sigma', h'(\beta)))] d\mu(\beta)$$

gives the minimum.

On account of the property that $D_{A,\Lambda}$ is bounded from below along with the completeness of $Slice(\mathcal{M}, g)_{(0,1)}$, some time-slicing s_0 of $(\mathcal{M}, g)_{(0,1)}$ is selected out w.r.t. the model $(\mathcal{M}', g')_{(0,1),s'}$, A (apparatus) and Λ (scale)⁶.

[2] *Assignment of a model to ‘reality’*

We can continue the same procedure for every model spacetime $\in \{models\}$ to choose the best-fitted model $(\mathcal{M}^*, g^*)_{(0,1),s^*}$ and, w.r.t. it, the time-slicing s_0^* of $(\mathcal{M}, g)_{(0,1)}$. Then one can regard $(\mathcal{M}_*, g_*)_{(0,1),s^*}$ to be the cosmological counterpart of $(\mathcal{M}, g)_{(0,1),s_0^*}$ w.r.t. (A, Λ) . In this way the spectral representation naturally leads us to the concept of *apparatus- and scale-dependent effective evolution of the universe*.

4 Example:(2+1)-dimensional flat spacetimes

As an illustration for the procedure in the previous section, let us consider a simple example. We choose as ‘reality’ the simplest (2+1)-dimensional flat spacetime with topology $T^2 \times \mathbf{R}$: We can construct such a spacetime from \mathbf{R}^3 by the identification in space, $(x + m, y + n) \sim (x, y)$, where $m, n \in \mathbf{Z}$. (Here, (x, y, t) is the standard coordinates for \mathbf{R}^3 .) We can imagine this spacetime as a static spacetime with a spatial section being a regular 2-torus (a torus constructed from a unit square by gluing the edges facing each other), if $t = const$ slicing is employed. Now, let $(\mathcal{M}, g)_{(0,1)}$ be a portion of the spacetime defined by $0 \leq t \leq 1$. Then $Slice_o(\mathcal{M}, g)_{(0,1)}$ denotes a set of all slices for the present $(\mathcal{M}, g)_{(0,1)}$, and $Slice(\mathcal{M}, g)_{(0,1)}$ is its completion.

As a set of model spacetimes, $\{models\}$, we take a set of all (2+1)-dimensional flat spacetimes of topology $T^2 \times \mathbf{R}$ with particular slices; For each model, a particular time-slicing is employed by which the line-element is represented as

$$ds^2 = -dt^2 + h_{ab}d\xi^a d\xi^b \quad ,$$

⁶ To be more precise, there can be more than one slicings that satisfy the condition. Furthermore, s_0 can be a limit point of $Slice_o(\mathcal{M}, g)_{(0,1)}$, viz. $s_0 \in Slice(\mathcal{M}, g)_{(0,1)} \setminus Slice_o(\mathcal{M}, g)_{(0,1)}$. In such a case, one would judge that $(\mathcal{M}', g')_{(0,1),s'}$ is not an appropriate model for $(\mathcal{M}, g)_{(0,1),s}$. In any case, it is desirable to investigate the mathematical structure of $Slice(\mathcal{M}, g)_{(0,1)}$ in more detail.

where

$$h_{ab} = \frac{V}{\tau^2} \begin{pmatrix} 1 & \tau^1 \\ \tau^1 & |\tau|^2 \end{pmatrix} \quad .$$

Here (τ^1, τ^2) are the Teichmüller parameters of a 2-torus, and $\tau := \tau^1 + i\tau^2$, $\tau^2 > 0$; (τ^1, τ^2) and $V(> 0)$ are functions of t only; The periodicity in the coordinates ξ^1 and ξ^2 with period 1 are understood. We note that (τ^1, τ^2) represent the shape of a parallelogram⁷ which forms the 2-torus by the edge-gluing; V represents the 2-volume of the 2-torus [12].

The functional forms for τ^1 , τ^2 and V are not arbitrary; The evolutions of τ^1 , τ^2 and V w.r.t. t are determined by a simple constrained Hamiltonian system [13] $\{(\tau^1, p_1), (\tau^2, p_2), (V, \sigma); H \simeq 0\}$. Thus, in this example, $\{models\}$ is parameterized by distinct initial conditions for the Hamiltonian system. In other words, 4 parameters are required in principle to characterize each model in $\{models\}$.

Now, take one model in $\{models\}$, and consider its portion characterized by $0 \leq t \leq 1$. This portion of the model corresponds to $(\mathcal{M}', g')_{(0,1),s'}$ in the previous section. We easily get the spectra for each time-slice Σ' of $(\mathcal{M}', g')_{(0,1),s'}$: The Laplacian in this case becomes $\Delta = h^{ab} \frac{\partial}{\partial \xi^a} \frac{\partial}{\partial \xi^b}$; The normalized eigenfunctions of the Laplacian are

$$f_{n_1 n_2}(\xi^1, \xi^2) = \exp(i2\pi n_1 \xi^1) \cdot \exp(i2\pi n_2 \xi^2)$$

with the spectra

$$\begin{aligned} \lambda'_{n_1 n_2} &= \frac{4\pi^2}{V\tau^2} |n_2 - \tau n_1|^2 \\ &= \frac{4\pi^2}{V\tau^2} (|\tau|^2 n_1^2 - 2\tau^1 n_1 n_2 + n_2^2) \quad (n_1, n_2 \in \mathbf{Z}) \quad . \end{aligned} \quad (5)$$

On the other hand, the ‘reality’ $(\mathcal{M}, g)_{(0,1)}$ is identical with an element of $\{models\}$ when $t = const$ slicing s_c is employed; viz. $(\mathcal{M}, g)_{(0,1),s_c}$ is identical with the model characterized by $\tau^1 \equiv 0$, $\tau^2 \equiv 1$ and $V \equiv 1$. Then, for every spatial section Σ of $(\mathcal{M}, g)_{(0,1),s_c}$, the spectra become

$$\lambda_{n_1 n_2} = 4\pi^2 (n_1^2 + n_2^2) \quad (n_1, n_2 \in \mathbf{Z}) \quad . \quad (6)$$

We can measure the spectral distance with the help of Eqs.(5) and (6) $d_N((\Sigma, h(t)), (\Sigma', h'(t)))$. It is obvious that $D_N((\mathcal{M}, g)_{(0,1),s}, (\mathcal{M}', g')_{(0,1),s'})$ gives the absolute minimum, 0, only when the model $(\mathcal{M}', g')_{(0,1),s'}$ is the one characterized by $\tau^1 \equiv 0$, $\tau^2 \equiv 1$ and $V \equiv 1$, and the slicing of the ‘reality’ is $s = s_c$.

⁷ In the present parametrization, the coordinates of four vetices of the parallelogram $OACB$ are $O = (0, 0)$, $A = (\tau^1/\sqrt{\tau^2}, \sqrt{\tau^2})$, $B = (1/\sqrt{\tau^2}, 0)$ and $C = (\frac{1+\tau^1}{\sqrt{\tau^2}}, \sqrt{\tau^2})$ [12].

5 Dynamics of spectra

We have established the spectral distance, which provides a basis for comparing the real spatial geometry with a model spatial geometry. We can now investigate the spectral distance between ‘reality’ and a model as a function of time, which serves as the quantitative analysis of the influence of the averaging procedure on the effective dynamics of the universe. Here we see the usefulness of the spectral distance: On one hand it has a nice mathematical properties, and on the other hand, it can be handled explicitly. Thus, we now need dynamical equations for the spectra.

We first prepare concise notations for specific integrals that appear frequently below. Let $A(\cdot)$ and $A_{ab}(\cdot)$ be any function and any symmetric tensor field, respectively, defined on a spatial section Σ . Let $\{f_n\}_{n=0}^\infty$ be the eigenfunctions of the Laplacian. Then we define

$$\begin{aligned}\langle A \rangle_{mn} &:= \int_{\Sigma} f_n A(x) f_m \ , \quad \langle A \rangle_n := \langle A \rangle_{nn} \ , \\ \langle A_{ab} \rangle_{mn} &:= \frac{1}{\sqrt{\lambda_m \lambda_n}} \int_{\Sigma} \partial^a f_m A_{ab}(x) \partial^b f_n \ .\end{aligned}$$

In order to derive the spectral evolution equations, we first recall a basic result of the perturbation theory (“Fermi’s golden rule”)

$$\delta \lambda_n = -\langle \delta \Delta \rangle_n \ . \quad (7)$$

Noting $\Delta f = \frac{1}{\sqrt{h}}(\sqrt{h} h^{ab} \partial_b f)_{,a}$, it is straightforward to get

$$\langle \delta \Delta \rangle_n = \langle \overline{\delta h_{ab}} \rangle_n \lambda_n + \frac{1}{2} \langle h \cdot \delta h \rangle_n \lambda_n \ , \quad (8)$$

where $h \cdot \delta h := h^{ab} \delta h_{ab}$ and $\overline{\delta h_{ab}} := \delta h_{ab} - \frac{1}{2} h \cdot \delta h h_{ab}$. Combining Eq.(8) with Eq.(7), simple manipulations lead to a formula [14]

$$\delta \lambda_n = -\langle \delta h_{ab} \rangle_n \lambda_n + \frac{1}{4} \langle \Delta(h \cdot \delta h) \rangle_n \ . \quad (9)$$

Now we identify δh_{ab} in Eq.(9) with the time-derivative of the spatial metric, \dot{h}_{ab} , w.r.t. the time-slicing: δh_{ab} should be replaced by $\dot{h}_{ab} = 2NK_{ab} + 2D_{(a}N_{b)}$. Here N and N_a are the lapse function and the shift vector, respectively; K_{ab} is the extrinsic curvature and $K := K_a^a$. After some manipulations [14], we finally reach the basic formula for the spectral evolution,

$$\dot{\lambda}_n = -2\langle NK_{ab} \rangle_n \lambda_n + \frac{1}{2} \langle \Delta(NK) \rangle_n \ . \quad (10)$$

We note that the shift vector N_a does not appear in the final result, Eq.(10). This result comes from the fact that the spectra are spatial diffeomorphism invariant quantities.

For simplicity, let us set $N \equiv 1$. Then we get

$$\dot{\lambda}_n = -\frac{2}{D-1}\langle K \rangle_n \lambda_n + \frac{D-3}{2(D-1)}\langle \Delta K \rangle_n - 2\langle \epsilon_{ab} \rangle_n \lambda_n, \quad (11)$$

where $\epsilon_{ab} := K_{ab} - \frac{1}{D-1}Kh_{ab}$, and D is the spacetime dimension ($D = 4$ in the ordinary case).

It is not our present aim to go into the detailed analysis on the dynamics of the spectra, which will be done elsewhere [14]. Here we only discuss a simple example as a demonstration of the usefulness of the spectral scheme: Let us investigate the scale-dependence of the effective Hubble parameter H_{eff} .

Suppose the spacetime is close to the closed Friedman-Robertson-Walker universe. In this case, ΔK and ϵ_{ab} are regarded as small. (More precisely, $\langle \Delta K \rangle_{mn}$ and $\langle \epsilon_{ab} \rangle_{mn}$ are small compared to $\langle K \rangle_{mn} \sqrt{\lambda_m \lambda_n}$ and $\langle K \rangle_{mn}$, respectively ($m, n = 1, 2, \dots$).) Then Eq.(11) can be written as (we set $D = 4$)

$$\dot{\lambda}_n = -\frac{2}{3}\langle K \rangle_n \left[1 - \frac{1}{4} \frac{\langle \Delta K \rangle_n}{\langle K \rangle_n \lambda_n} + 3 \frac{\langle \epsilon_{ab} \rangle_n}{\langle K \rangle_n} \right] \lambda_n, \quad (12)$$

hence

$$(H_{eff})_n = \frac{1}{3}\langle K \rangle_n \left[1 - \frac{1}{4} \frac{\langle \Delta K \rangle_n}{\langle K \rangle_n \lambda_n} + 3 \frac{\langle \epsilon_{ab} \rangle_n}{\langle K \rangle_n} \right]. \quad (13)$$

The last two terms in the bracket describe the influence of inhomogeneity and anisotropy on H_{eff} at scale Λ . Here we note once more that the spectral representation naturally describes the apparatus- and the scale-dependent picture of the universe (in the present example, $(H_{eff})_n$).

6 Spacetime picture from the viewpoint of the spectral representation

We have discussed the spectral representation of geometrical structures in connection with the averaging problem in cosmology. In particular we have introduced \mathcal{S}_N , the space of all spaces equipped with the spectral distance, and have shown that \mathcal{S}_N possesses several desirable properties as a basic arena for spacetime physics.

We have sketched the model-fitting procedure in the framework of the spectral representation, and have also suggested how to analyze the dynamical aspects of the averaging procedure within this framework. These arguments imply that the spectral scheme seems to be suitable for the analysis of the averaging problem. In fact, it naturally describes the apparatus- and scale-dependent effective evolution of the universe.

Finally, let us briefly discuss how spaces look like from the viewpoint of the spectral representation.

One may imagine the whole of the geometrical information of a space as a collection of all spectra such as

$$Space = \bigcup_i (\mathcal{D}_i, \{\lambda_n^{(i)}\}_{n=0}^\infty, \{f_n^{(i)}\}_{n=0}^\infty) \quad ,$$

where \mathcal{D}_i denotes an elliptic operator and $\{\lambda_n^{(i)}\}_{n=0}^\infty$ and $\{f_n^{(i)}\}_{n=0}^\infty$ are its spectra and the eigenfunctions, respectively. The index i runs over all possible elliptic operators. A single observation is related to a subclass of elliptic operators corresponding to the observational apparatus. Thus we get only a small portion of the whole geometrical information of the space by a single observation. Such incomplete information may not be enough to determine geometry uniquely. Only one has to do then is to conduct other kinds of observation corresponding to other types of elliptic operators in order to get further information on geometry. (This is the physical interpretation of ‘isospectral manifolds’, viz. non-isometric manifolds with the identical spectra of the Laplacian.) It is also tempting to regard the spectral information more fundamental than the concept of Riemannian manifolds. Further investigations are needed to judge to what extent such a viewpoint of spacetime geometry is valid.

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